SOLUTION TO THE KHAVINSON PROBLEM NEAR THE BOUNDARY OF THE UNIT BALL

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ABSTRACT. This paper deals with an extremal problem for harmonic functions in the unit ball of $\mathbf{R}^n$. We are concerned with the pointwise sharp estimates for the gradient of real–valued bounded harmonic functions. Our main result may be formulated as follows. The sharp constants in the estimates for the absolute value of the radial derivative and the modulus of the gradient of a bounded harmonic function coincide near the boundary of the unit ball. This result partially confirms a conjecture posed by D. Khavinson.

1. INTRODUCTION

1.1. **Gradient estimates in plane domains.** For $w = (w_1, w_2)$ which belongs to the upper half–plane $\mathbf{R}^2_+ = \{y = (y_1, y_2) \in \mathbf{R}^2 : y_2 > 0\}$ the gradient estimate

$$|\nabla U(w)| \le \frac{2}{\pi}\frac{1}{w_2}\sup_{y\in\mathbf{R}^2_+}|U(y)| \tag{1.1}$$

is sharp, if one assumes that $U(y)$ is among bounded harmonic functions in $\mathbf{R}^2_+$. Using the conformal transformation of the unit disc $\mathbf{B}^2 = \{x \in \mathbf{R}^2 : |x| < 1\}$ onto $\mathbf{R}^2_+$ given by $w = i(1+z)/(1-z)$, one easily transfers (1.1) into the following pointwise optimal estimate

$$|\nabla U(z)| \le \frac{4}{\pi}\frac{1}{1-|z|^2}\sup_{x\in\mathbf{B}^2}|U(x)|, \tag{1.2}$$

where this time $U(x)$ is a bounded harmonic function in the unit disc, and $z \in \mathbf{B}^2$ is arbitrary. For the above inequality we refer to [5, 9]; see also Chapter 4 in [10].

The inequality (1.2) may be viewed as a harmonic analogy of the classical Schwarz lemma for bounded analytic functions

$$|f'(z)| \le \frac{1}{1-|z|^2}\sup_{w\in\mathbf{B}^2}|f(w)|.$$

The famous Schwarz–Pick lemma improves the preceding inequality for analytic functions which map the unit disc onto itself.

As a recent result there is also a Schwarz–Pick type inequality for harmonic functions which send the unit disc $\mathbf{B}^2$ into the interval $(-1, 1)$. For this result see the paper of D. Kalaj and M. Vuorinen [8], where the authors also give a counterexample which shows that one cannot expect a Schwarz–Pick type inequality for harmonic mappings of the unit disc

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onto itself without additional assumptions. More detailed, the inequality (1.2) is improved in a new direction as

$$|\nabla U(z)| \leq \frac{4}{\pi}\frac{1-U(z)^2}{1-|z|^2} \tag{1.3}$$

for real–valued harmonic functions bounded by 1 in $\mathbf{B}^2$. The main ingredients in the proof of (1.3) are the classical Schwarz–Pick lemma, certain conformal transformations, and the fact that every harmonic function in a simply–connected plane domain is a real part of an analytic function. The extremal functions for (1.3) are given in [17]. The inequality (1.3) is equivalent to the following one $d_h(U(z),U(w)) \leq 4/\pi d_h(z,w)$ for $z,\, w \in \mathbf{B}^2$; here d_h stands for the hyperbolic distance in the unit disc $\mathbf{B}^2$. Therefore, real–valued harmonic functions bounded by 1 are Lipschitz continuous with respect to the hyperbolic metric with the optimal Lipschitz constant equal to $4/\pi$. On the other hand, the inequality (1.2) says that a harmonic function $U : \mathbf{B}^2 \to (-1,1)$ is Lipschitz continuous regarding the Euclidean distance d_e and the hyperbolic distance d_h, respectively, i.e., $d_e(U(z),U(w)) \leq 4/\pi d_h(z,w)$. We would like to refer to the F. Colonna paper [5] where the preceding inequality is considered for harmonic mappings of $\mathbf{B}^2$ onto itself along with the extremal problem

$$\sup_{z\neq w}\frac{d_e(U(z),U(w))}{d_h(z,w)} = \frac{4}{\pi}.$$

The estimate (1.2) is also true in this case with the norm of the differential of $U(z)$ on the left side.

In the reference [10] inequalities similar to (1.1) and (1.2) are considered in the context of the so called real–part theorems for analytic functions. To make a connection, let a function $f(z)$ be analytic in the unit disc, and let its real part $\mathrm{Re} f(z)$ be bounded there. Since $|\nabla \mathrm{Re} f(z)| = |f'(z)|$, one may rewrite (1.2) as the following estimate for the first derivative

$$|f'(z)| \leq \frac{4}{\pi}\frac{1}{1-|z|^2}\sup_{w\in\mathbf{B}^2}|\mathrm{Re} f(w)|,$$

which is known as the Lindelöf inequality in the unit disc. The classical work of A. J. Macintyre and W. W. Rogosinski [16] contains in some cases explicit pointwise sharp estimates for modulus of derivatives of analytic functions with $\sup_{w\in\mathbf{B}^2}|f(w)|$ on the right side.

1.2. **The Khavinson problem.** The question arises as to what may be done for bounded real–valued harmonic functions in domains in $\mathbf{R}^n$. Let us first precise the notation in high–dimensional settings. For $x = (x_1,\ldots,x_{n-1},x_n) \in \mathbf{R}^n$ denote $x' = (x_1,\ldots,x_{n-1}) \in \mathbf{R}^{n-1}$. Let $\mathbf{R}^n_+ = \{(x',x_n) \in \mathbf{R}^n : x_n > 0\}$ be the upper half–space, and let $\mathbf{B}^n = \{x \in \mathbf{R}^n : |x| < 1\}$ be the unit ball in $\mathbf{R}^n$. We denote by $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ the standard base in $\mathbf{R}^n$. In the rest of the paper we assume that $n > 2$.

Here and henceforth $\Gamma(z)$ denotes the Gamma function. Recently G. Kresin and V. Maz'ya [12] proved that

$$|\nabla U(x)| \leq \frac{4}{\sqrt{\pi}}\frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\frac{1}{x_n}\sup_{y\in\mathbf{R}^n_+}|U(y)|, \tag{1.4}$$

where $U(y)$ is a (real–valued) bounded harmonic function in $\mathbf{R}^n_+$, and $x = (x',x_n) \in \mathbf{R}^n_+$. This pointwise optimal gradient estimate, which generalizes the half–plane result stated in the beginning of the paper, arises in result of solution of the Khavinson type problem in the half–space.

In order to formulate the Khavinson problem and the corresponding conjecture, we introduce the notation we need; we will mostly follow [11, 12]. For every x in the unit ball $\mathbf{B}^n$ (in the half–space $\mathbf{R}^n_+$) and for every $\ell \in \partial\mathbf{B}^n$ let $\mathcal{C}(x)$ and $\mathcal{C}(x;\ell)$ denote, respectively, the optimal number for the gradient estimate

$$|\nabla U(x)| \le \mathcal{C}(x) \sup_y |U(y)|, \tag{1.5}$$

and the optimal number for the gradient estimate in the direction ℓ, i.e., the smallest number such that

$$|\langle \nabla U(x), \ell\rangle| \le \mathcal{C}(x;\ell) \sup_y |U(y)|, \tag{1.6}$$

where $U(y)$ is among bounded harmonic functions in $\mathbf{B}^n$ (in $\mathbf{R}^n_+$). Although we use the same notation for the gradient estimates in two different settings, we believe that this will not cause any confusion. We will occasionally use the standard notation $\partial U(x)/\partial \ell$ for $\langle \nabla U(x), \ell\rangle$.

It is especially important for us the normal direction accompanied to a point in a considered domain. We use the notation $\mathbf{n}_x$ for the normal direction at the given point x. In the unit ball setting for $x \ne 0$ the direction $\mathbf{n}_x$ is the outward unit vector orthogonal to the boundary of the unit ball $|x|\mathbf{B}^n$; in this case we have $\mathbf{n}_x = x/|x|$, i.e., $\mathbf{n}_x$ coincides with the radial direction. In the half–space setting the normal direction $\mathbf{n}_x$ is the outward unit vector orthogonal to the boundary of the half–space $(0, x_n) + \mathbf{R}^n_+$. Obviously, $\mathbf{n}_x = -\mathbf{e}_n$ for every $x \in \mathbf{R}^n_+$.

Since

$$|\nabla U(x)| \ = \sup_{\ell \in \partial \mathbf{B}^n} |\langle \nabla U(x), \ell\rangle| \, ,$$

we clearly have

$$\mathcal{C}(x) \ = \sup_{\ell \in \partial \mathbf{B}^n} \mathcal{C}(x;\ell). \tag{1.7}$$

It turned out that the optimisation problem on the right side of (1.7) is very difficult, especially if one considers harmonic functions in the unit ball. The Khavinson conjecture and its analogue for the half–space claim that

Conjecture 1.1 (Cf. [11, 12])**.**

$$\mathcal{C}(x) = \mathcal{C}(x;\mathbf{n}_x).$$

We briefly review the background of the above conjecture. In 1992, D. Khavinson [9] found the sharp coefficient in the pointwise estimate for the absolute value of the radial derivative of a bounded harmonic function in the ball $\mathbf{B}^3$. He made a conjecture that the same coefficient should be appear in the stronger pointwise sharp estimate for the modulus of the gradient of a bounded harmonic function in $\mathbf{B}^3$. In the recent papers by G. Kresin and V. Maz'ya [11, 12] this problem and its analogue in the half–space were considered in more general aspect. In particular, in [12] they proved the above conjecture for the upper half–space, and, as a consequence, found the sharp coefficient in the inequality (1.4). For more information about the Khavinson problem we refer to Chapter 6 in [14].

The rest of the paper contains two more sections. In the following one we obtain various integral representations of the coefficients $\mathcal{C}(x;\ell)$ for the unit ball. Particularly, the representation which is given in Theorem 2.12 is very important for our considerations. By this theorem we have $\mathcal{C}(x;\ell_1) = \mathcal{C}(x;\ell_2)$ if the angle between the straight lines $L_{\mathbf{n}_x}$ and L_{ℓ_1} is equal to the angle between $L_{\mathbf{n}_x}$ and L_{ℓ_2}. Recall that the angle between the straight lines $L_{\ell_1} = \{\lambda\ell_1 : \lambda \in \mathbf{R}\}$ and $L_{\ell_2} = \{\mu\ell_2 : \mu \in \mathbf{R}\}$, determined by the unit vectors ℓ_1 and

ℓ_2, respectively, is $\arccos|\langle \ell_1, \ell_2\rangle|$. The main aim of this paper is to treat the optimisation problem in (1.7) in the unit ball setting. This is done in the third section. At this moment we are not able to prove that the above stated conjecture is true, but we prove that the statement of the conjecture holds when $x \in \mathbf{B}^n$ is near the boundary of the unit ball, i.e., if $|x| \approx 1$. Therefore, in the gradient estimate (1.5) one can replace $\mathcal{C}(x)$ with $\mathcal{C}(x; \mathbf{n}_x)$ (as it is possible in the half–space setting for all $x \in \mathbf{R}^n_+$), if $|x|$ is near 1. On the other hand, it seems that the problem in (1.7) is much more harder when $|x|$ is close to 0. The problem is trivial for $x = 0$ (Remark 2.2). Moreover, according to our main result, we may conclude that $\mathcal{C}(x) = \mathcal{C}(x; \ell)$ only for $\ell = \pm \mathbf{n}_x$, if $|x| \approx 1$.

In the half–space setting it is possible to obtain the factorization $\mathcal{C}(x; \ell) = x_n^{-1} C(\ell)$, where $C(\ell)$ does not depend on $x = (x', x_n) \in \mathbf{R}^n_+$. This is proved in [12], where the integral expression representing $C(\ell)$ is also obtained. The integral representation of $C(\ell)$ also appears in this paper in the second section (see Proposition 2.16 there). The main difference between the half–space case and the unit ball case is that in the later case one cannot represent $\mathcal{C}(x; \ell)$ in the form which is a number depending only on ℓ multiplied by the asymptotic factor $(1 - |x|)^{-1}$, as it is possible in the first case; this is the content of Lemma 2.3 where it is proved that $\mathcal{C}(x; \ell) = (1 - |x|)^{-1} C(x; \ell)$, and $C(x; \ell)$ is bounded as a function of two variables.

If U is a bounded harmonic function in $\mathbf{B}^n$, then

$$V(x) = |x + \mathbf{e}_n/2|^{2-n} U(T(x)),$$

is a such one function in $\mathbf{R}^n_+$, where $T : \mathbf{R}^n_+ \to \mathbf{B}^n$ is the Möbius transform given by

$$T(x) = \frac{|x + \mathbf{e}_n/2|}{|x + \mathbf{e}_n/2|^2} - \mathbf{e}_n.$$

It seems that this transformation does not provide any fruitful connection between the coefficients $\mathcal{C}(x; \ell)$ in the two different settings. However, it happens that

$$C(\ell') = \lim_{\rho \to 1^-} C(\rho \mathbf{e}_1; \ell),$$

where ℓ' and ℓ are the directions such that the angles between the corresponding normal directions are equal. This limit relation along with the approach of G. Kresin and G. Maz'ya from [12] is crucial for our partial solution to the optimisation problem.

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2. Integral representations of $\mathcal{C}(x; \ell)$ and $C(x; \ell)$

2.1. **Representations as integrals over the unit sphere.** Finding some symmetries and relations between the numbers $\mathcal{C}(x; \ell)$ itself is very useful in obtaining the integral representations of $\mathcal{C}(x; \ell)$. First of all we will prove

Lemma 2.1. *For every $x \in \mathbf{B}^n$ and $\ell \in \partial \mathbf{B}^n$ we have*

$$\mathcal{C}(\mathcal{A}x; \mathcal{A}\ell) = \mathcal{C}(x; \ell),$$

where $\mathcal{A} \in O(n)$.

In the above lemma $O(n)$ is the group of all orthogonal transformations of $\mathbf{R}^n$. It is well known that the group $O(n)$ preserves harmonic functions in the unit ball; for example, see Chapter 1 in [3].

Proof. Let $\mathcal{A}$ be an orthogonal transformation of $\mathbf{R}^n$. We have

$$\begin{aligned}\mathcal{C}(\mathcal{A}x;\mathcal{A}\ell) &= \sup_{U,\, \|U\|_\infty=1} |\langle \nabla U(\mathcal{A}x), \mathcal{A}\ell\rangle| \\ &= \sup_{U,\, \|U\|_\infty=1} |\langle \mathcal{A}\nabla U(\mathcal{A}x), \ell\rangle| \quad (\text{since } \mathcal{A}^* = \mathcal{A}) \\ &= \sup_{U,\, \|U\|_\infty=1} |\langle \nabla (U\circ\mathcal{A})(x), \ell\rangle| \\ &= \sup_{V,\, \|V\|_\infty=1} |\langle \nabla V(x), \ell\rangle| = \mathcal{C}(x;\ell),\end{aligned}$$

which is the desired relation. The notation $\|\cdot\|_\infty$ used above stands for the essential supremum of a function defined on a measure space. □

Remark 2.2. If we take $x = 0$ in Lemma 2.1 we obtain $\mathcal{C}(0;\mathcal{A}\ell) = \mathcal{C}(0;\ell)$, for every $\mathcal{A} \in O(n)$ and every $\ell \in \partial\mathbf{B}^n$. Regarding the fact that the group $O(n)$ acts transitively on spheres with center in $0 \in \mathbf{R}^n$, this implies that $C(0;\ell)$ does not depend on the choice of a direction.

The following lemma isolates the asymptotic factor from $\mathcal{C}(x;\ell)$, and gives the first integral representation of $\mathcal{C}(x;\ell)$ as an integral over the unit sphere $\partial\mathbf{B}^n$.

Lemma 2.3. *For every $x \in \mathbf{B}^n$ and $\ell \in \partial\mathbf{B}^n$ we have*

$$\mathcal{C}(x;\ell) = (1-|x|)^{-1}C(x;\ell),$$

where we have denoted

$$C(x;\ell) = \frac{n}{1+|x|}\int_{\partial\mathbf{B}^n} \left|\left\langle \eta - \frac{n-2}{n}x, \ell\right\rangle\right| |\eta - x|^{2-n}\, d\sigma(\eta);$$

$d\sigma$ stands for the normalized area measure on the unit sphere $\partial\mathbf{B}^n$.

Remark 2.4. For $n = 2$ from this lemma it is not hard to calculate that

$$C(x;\ell) = \frac{4}{\pi}\frac{1}{1-|x|}$$

for all $x \in \mathbf{B}^2$ and $\ell \in \partial\mathbf{B}^2$. Therefore,

$$\mathcal{C}(x) = \mathcal{C}(x;\ell) = \frac{4}{\pi}\frac{1}{1-|x|^2}.$$

This gives the pointwise sharp gradient estimate (1.2). For more plane results we refer to [7].

Proof of Lemma 2.3. Let $U(y)$ be a bounded harmonic function in the unit ball $\mathbf{B}^n$. It has a radial boundary value $U^*(\zeta) = \lim_{r\to 1^-} U(r\zeta)$ for almost every $\zeta \in \partial\mathbf{B}^n$ (we assume the area measure on $\partial\mathbf{B}^n$), and $U(y)$ may be represented as the Poisson integral

$$U(y) = \mathrm{P}[U^*](y) = \int_{\partial\mathbf{B}^n} P(y,\zeta)U^*(\zeta)d\sigma(\zeta),$$

where

$$P(y,\zeta) = \frac{1-|y|^2}{|y-\zeta|^n}$$

is the Poisson kernel. One says that $U(y)$ is the Poisson extension of $U^*(\zeta)$. Moreover, there holds $\|U\|_\infty = \|U^*\|_\infty$. Because of the last relation, P acts as an isometric isomorphism between the space of bounded harmonic functions in $\mathbf{B}^n$ and the space of essentially bounded functions on the unit sphere $\partial\mathbf{B}^n$. For all these facts we refer to Chapter 6 in [3].

For $x \in \mathbf{B}^n$ and $\ell \in \partial\mathbf{B}^n$ we have

$$\langle \nabla U(x), \ell\rangle = \int_{\partial\mathbf{B}^n} \langle \nabla P(x,\zeta), \ell\rangle\, U^*(\zeta) d\sigma(\zeta). \tag{2.1}$$

Let Λ_ℓ denote the functional given on the left side of the above equation. It operates on the space of bounded harmonic functions in the unit ball, and we clearly have $\|\Lambda_\ell\| = \mathcal{C}(x;\ell)$. On the other hand, in view of the isometric isomorphism via the Poisson extension P between the space of all bounded harmonic functions in $\mathbf{B}^n$ and $L^\infty(\partial\mathbf{B}^n)$, the functional Λ_ℓ may be identified with the functional on $L^\infty(\partial\mathbf{B}^n)$ given by the right side of the equality (2.1). Therefore,

$$\mathcal{C}(x;\ell) = \int_{\partial\mathbf{B}^n} |\langle \nabla P(x,\zeta), \ell\rangle|\, d\sigma(\zeta). \tag{2.2}$$

Let $T_x(y)$ be the Möbius transform

$$T_x(y) = \frac{(1-|x|^2)(y-x) - |y-x|^2 x}{[y,x]^2},$$

where

$$[y,x] = |y||y^* - x|, \quad y^* = \frac{y}{|y|^2}.$$

The mapping T_x transforms the unit sphere $\partial\mathbf{B}^n$ onto itself bijectively. Moreover, when restricted to the unit sphere it takes the following form

$$T_x(\eta) = (1-|x|^2)\frac{\eta - x}{|\eta - x|^2} - x;$$

note that $[\eta, x] = |\eta - x|$ for $\eta \in \partial\mathbf{B}^n$, since $\eta^* = \eta$. We refer to the Ahlfors book [1] for a detailed survey of this class of important mappings.

We will take the change of variable

$$\zeta = -T_x(\eta), \quad \eta \in \partial\mathbf{B}^n$$

in the integral representation of $\mathcal{C}(x;\ell)$ given in (2.2). First of all one immediately finds

$$\nabla P(x,\zeta) = \frac{-2x|x-\zeta|^2 - n(1-|x|^2)(x-\zeta)}{|x-\zeta|^{n+2}}. \tag{2.3}$$

Since

$$x - \zeta = (1-|x|^2)\frac{\eta - x}{|\eta - x|^2}, \quad |x-\zeta| = \frac{1-|x|^2}{|\eta - x|},$$

we obtain

$$\nabla P(x,\zeta) = \left(\frac{-2x(1-|x|^2)^2}{|\eta - x|^2} - n\frac{(1-|x|^2)^2(\eta - x)}{|\eta - x|^2}\right) : \frac{(1-|x|^2)^{n+2}}{|\eta - x|^{n+2}},$$

which implies

$$\nabla P(x,\zeta) = ((n-2)x - n\eta) : \frac{(1-|x|^2)^n}{|\eta - x|^n}.$$

Since

$$d\sigma(\zeta) = \frac{(1-|x|^2)^{n-1}}{|\eta - x|^{2n-2}} d\sigma(\eta)$$

(see the Ahlfors book), regarding the above expression for $\nabla P(x,\zeta)$, we have

$$\langle \nabla P(x,\zeta), \ell\rangle\, d\sigma(\zeta) = -(1-|x|^2)^{-1}\, \langle n\eta - (n-2)x, \ell\rangle\, |\eta - x|^{2-n}\, d\sigma(\eta).$$

The preceding facts imply

$$\begin{aligned}\mathcal{C}(x;\ell) &= \int_{\partial\mathbf{B}^n} |\langle \nabla P(x,\zeta), \ell\rangle|\, d\sigma(\zeta)\\ &= (1-|x|^2)^{-1}\int_{\partial\mathbf{B}^n} |\langle n\eta-(n-2)x,\ell\rangle|\,|\eta-x|^{2-n}\, d\sigma(\eta)\\ &= (1-|x|)^{-1}C(x;\ell).\end{aligned}$$

Recall that in the formulation of this lemma we have denoted

$$C(x;\ell) = \frac{n}{1+|x|}\int_{\partial\mathbf{B}^n} \left|\left\langle \eta - \frac{n-2}{n}x, \ell\right\rangle\right| |\eta-x|^{2-n}\, d\sigma(\eta).$$

We have achieved what we wanted to prove. $\square$

Remark 2.5. Since $C(x;\ell) = (1-|x|)\mathcal{C}(x;\ell)$ it follows the first lema of this section that

$$C(\mathcal{A}x;\mathcal{A}\ell) = C(x;\ell)$$

for all orthogonal transformations $\mathcal{A}$. This also may be seen directly from the just obtained expression for $C(x;\ell)$ (similarly as in the following lemma).

Lemma 2.6. *Let $\mathcal{A}\in O(n)$ be a such transformation that $\mathcal{A}x = x$. Then*

$$C(x;\ell) = C(x;\mathcal{A}\ell)$$

for every $\ell\in\partial\mathbf{B}^n$.

Proof. Since $\mathcal{A}^{-1} = \mathcal{A}^*$, we also have $\mathcal{A}^*x = x$. It follows

$$|\mathcal{A}^*\eta - x| = |\mathcal{A}^*(\eta-x)| = |\eta-x|,$$

and

$$\left\langle \mathcal{A}^*\eta - \frac{n-2}{n}x, \ell\right\rangle = \left\langle \mathcal{A}^*\left(\eta - \frac{n-2}{n}x\right), \ell\right\rangle = \left\langle \eta - \frac{n-2}{n}x, \mathcal{A}\ell\right\rangle.$$

In the integral representation of $C(x;\ell)$ in Lemma 2.3 we will perform the change of variable: $\mathcal{A}^*\eta$ instead of η. Since the measure $d\sigma$ is rotation–invariant, we have

$$\begin{aligned}C(x;\ell) &= \frac{n}{1+|x|}\int_{\partial\mathbf{B}^n} \left|\left\langle \mathcal{A}^*\eta - \frac{n-2}{n}x, \ell\right\rangle\right| |\mathcal{A}^*\eta-x|^{2-n}\, d\sigma(\eta)\\ &= \frac{n}{1+|x|}\int_{\partial\mathbf{B}^n} \left|\left\langle \eta - \frac{n-2}{n}x, \mathcal{A}\ell\right\rangle\right| |\eta-x|^{2-n}\, d\sigma(\eta) = C(x;\mathcal{A}\ell),\end{aligned}$$

which we need. $\square$

Lemma 2.7. *If the angle between $\mathbf{n}_x$ and ℓ is equal to the angle between $\mathbf{n}_x$ and ℓ', then*

$$C(x;\ell) = C(x;\ell').$$

Proof. Since $\langle \mathbf{n}_x, \ell\rangle = \langle \mathbf{n}_x, \ell'\rangle$, there exists an orthogonal transformation $\mathcal{A}$ such that $\mathcal{A}\mathbf{n}_x = \mathbf{n}_x$ (which is the same condition as $\mathcal{A}x = x$) and $\mathcal{A}\ell = \ell'$. It view of Lemma 2.6 the statement of the current one follows. $\square$

2.2. **Representations as double integrals.** Let ω_n denote the area of the unit sphere in $\mathbf{R}^n$. It is well known that

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

We set $\omega_1 = 2$, which is consistent with the above formula.

For $\tau \in [0, 2\pi]$ let

$$\ell_\tau = \cos\tau \mathbf{e}_1 + \sin\tau \mathbf{e}_2.$$

In this subsection we will restrict our attention to the numbers $C(\rho\mathbf{e}_1; \ell_\tau)$, where $0 \le \rho \le 1$ and $0 \le \tau \le 2\pi$.

Lemma 2.8. *For every $\rho \in [0, 1]$ and $\tau \in [0, 2\pi]$ we have*

$$C(\rho\mathbf{e}_1; \ell_\tau) = \frac{2\omega_{n-2}}{\omega_n}\frac{1}{1+\rho}\int_0^\pi \mathcal{R}(\tilde\vartheta)\sin^{n-2}\tilde\vartheta d\tilde\vartheta \int_0^\pi |\tilde{\mathcal{G}}(\varphi, \vartheta, \tau)| \sin^{n-3}\varphi d\varphi,$$

where

$$\tilde{\mathcal{G}}(\varphi, \tilde\vartheta, \tau) = \frac{n}{2}(\cos\tilde\vartheta - \alpha_\rho)\cos\tau + \frac{n}{2}(\sin\tilde\vartheta\cos\varphi)\sin\tau, \quad \alpha_\rho = \frac{n-2}{n}\rho,$$

and

$$\mathcal{R}(\tilde\vartheta) = (1 + \rho^2 - 2\rho\cos\tilde\vartheta)^{1-\frac{n}{2}}.$$

Proof. In the integral representation of $C(\rho\mathbf{e}_1; \ell_\tau)$ in Lemma 2.3:

$$C(\rho\mathbf{e_1}; \ell_\tau) = \frac{n}{1+\rho}\int_{\partial\mathbf{B}^n}\left|\left\langle \eta - \frac{n-2}{n}\rho\mathbf{e_1}, \ell_\tau\right\rangle\right| |\eta - \rho\mathbf{e_1}|^{2-n} d\sigma(\eta)$$

we will take the change of variable introducing the spherical coordinates in the following way $\eta = (\cos\tilde\vartheta, \sin\tilde\vartheta\cos\varphi, \dots)$. The range of $\tilde\vartheta$ is $[0, \pi]$. If $n > 3$, the range of φ is $[0, \pi]$; if $n = 3$, then the range of φ is $[0, 2\pi]$.

Straightforward calculations yield

$$|\langle \eta - \alpha_\rho\mathbf{e}_1, \ell_\tau\rangle| = |(\cos\tilde\vartheta - \alpha_\rho)\cos\tau + (\sin\tilde\vartheta\cos\varphi)\sin\tau|$$

and

$$|\eta - \rho\mathbf{e}_1|^{2-n} = (1 + \rho^2 - 2\rho\cos\tilde\vartheta)^{1-\frac{n}{2}}.$$

Assume first that $n > 3$. Then we have

$$\begin{aligned}
C(\rho\mathbf{e}_1; \ell_\tau) &= \frac{n}{1+\rho}\int_{\partial\mathbf{B}^n} |\langle \eta - \alpha_\rho\mathbf{e}_1, \ell_\tau\rangle| \, |\eta - \rho\mathbf{e}_1|^{2-n} d\sigma(\eta) \\
&= \frac{2\omega_{n-2}}{\omega_n}\frac{1}{1+\rho}\int_0^\pi\int_0^\pi |\tilde{\mathcal{G}}(\varphi, \tilde\vartheta, \tau)|\mathcal{R}(\tilde\vartheta)\sin^{n-2}\tilde\vartheta\sin^{n-3}\varphi d\tilde\vartheta d\varphi \\
&= \frac{2\omega_{n-2}}{\omega_n}\frac{1}{1+\rho}\int_0^\pi \mathcal{R}(\tilde\vartheta)\sin^{n-2}\tilde\vartheta d\tilde\vartheta\int_0^\pi |\tilde{\mathcal{G}}(\varphi, \vartheta, \tau)| \sin^{n-3}\varphi d\varphi.
\end{aligned}$$

The obtained expression for $C(\rho\mathbf{e}_1; \ell_\tau)$ is also valid if $n = 3$, but in this case we must bear in mind one more transformation of the inner integral above where the integration is

against φ. This step is

$$\begin{aligned}\int_\pi^{2\pi} |\tilde{\mathcal{G}}(\varphi,\tilde\vartheta,\tau)|d\varphi &= \int_0^\pi |\tilde{\mathcal{G}}(\pi+\varphi,\tilde\vartheta,\tau)|d\varphi\\ &= \int_0^\pi |\tilde{\mathcal{G}}(\pi-\varphi,\tilde\vartheta,\tau)|d\varphi\\ &= \int_0^\pi |\tilde{\mathcal{G}}(\varphi,\tilde\vartheta,\tau)|d\varphi\end{aligned}$$

(we have introduced φ instead of $\pi-\varphi$ in the last integral). □

Using some trigonometric transformations we will now prove

Lemma 2.9. *For every $\rho\in[0,1]$ and $\tau\in[0,2\pi]$ we have*

$$C(\rho\mathbf{e}_1;\ell_\tau) = \frac{4\omega_{n-2}}{\omega_n}\frac{1}{1+\rho}\int_0^\pi \sin^{n-3}\varphi d\varphi\int_0^{\frac{\pi}{2}} |\mathcal{G}(\varphi,\vartheta,\tau)|\mathcal{S}(\vartheta)d\vartheta,$$

where we have denoted

$$\mathcal{G}(\varphi,\vartheta,\tau) = (n\cos^2\vartheta-\beta_\rho)\cos\tau + (n\sin\vartheta\cos\vartheta\cos\varphi)\sin\tau,\quad \beta_\rho = \frac{n-(n-2)\rho}{2},$$

and

$$\mathcal{S}(\vartheta) = \frac{\sin^{n-2}2\vartheta}{((1+\rho)^2-4\rho\sin^2\vartheta)^{\frac{n}{2}-1}}.$$

Proof. Interchanging the order of integration, from the preceding lemma we immediately obtain

$$C(\rho\mathbf{e}_1;\ell_\tau) = \frac{2\omega_{n-2}}{\omega_n}\frac{1}{1+\rho}\int_0^\pi \sin^{n-3}\varphi d\varphi\int_0^\pi |\tilde{\mathcal{G}}(\varphi,\tilde\vartheta,\gamma)|\mathcal{R}(\tilde\vartheta)\sin^{n-2}\tilde\vartheta d\tilde\vartheta,$$

where

$$\tilde{\mathcal{G}}(\varphi,\tilde\vartheta,\tau) = \frac{n}{2}(\cos\tilde\vartheta-\alpha_\rho)\cos\tau + \frac{n}{2}(\sin\tilde\vartheta\cos\varphi)\sin\tau,$$

and

$$\mathcal{R}(\tilde\vartheta) = (1+\rho^2-2\rho\cos\tilde\vartheta)^{1-\frac{n}{2}}.$$

Introduce $\vartheta=\tilde\vartheta/2$. We transform

$$\begin{aligned}|\tilde{\mathcal{G}}(\varphi,\tilde\vartheta,\tau)| &= \frac{n}{2}|(\cos\tilde\vartheta-\alpha_\rho)\cos\tau + (\sin\tilde\vartheta\cos\varphi)\sin\tau|\\ &= \frac{n}{2}|(1-2\sin^2\vartheta-\alpha_\rho)\cos\tau + (2\sin\vartheta\cos\vartheta\cos\varphi)\sin\tau|\\ &= |(n\sin^2\vartheta - n(1-\alpha_\rho)/2)\cos\tau - (n\sin\vartheta\cos\vartheta\cos\varphi)\sin\tau|\\ &= |(n\sin^2\vartheta-\beta_\rho)\cos\tau - (n\sin\vartheta\cos\vartheta\cos\varphi)\sin\tau|.\end{aligned}$$

In the last equality we have used the substitution

$$\frac{n(1-\alpha_\rho)}{2} = \frac{n-(n-2)\rho}{2} = \beta_\rho.$$

In the integral representation of $C(\rho\mathbf{e}_1;\ell_\tau)$ given above we will write $\pi-\varphi$ instead of φ, and $\pi/2-\vartheta$ instead of ϑ (i.e. $\tilde\vartheta/2$). In this way we obtain

$$C(\rho\mathbf{e}_1;\ell_\tau) = \frac{4\omega_{n-2}}{\omega_n}\frac{1}{1+\rho}\int_0^\pi \sin^{n-3}\varphi d\varphi\int_0^{\frac{\pi}{2}} |\mathcal{G}(\varphi,\vartheta,\tau)|\mathcal{S}(\vartheta)d\vartheta,$$

where

$$\begin{aligned}\mathcal{G}(\varphi,\vartheta,\tau) &= (n\sin^2(\pi/2-\vartheta)-\beta_\rho)\cos\tau - (n\sin\vartheta\cos\vartheta\cos(\pi-\varphi))\sin\tau\\ &= (n\cos^2\vartheta-\beta_\rho)\cos\tau + (n\sin\vartheta\cos\vartheta\cos\varphi)\sin\tau,\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}(\vartheta) &= \sin^{n-2}2(\pi/2-\vartheta)\mathcal{R}(2(\pi/2-\vartheta)) = \frac{\sin^{n-2}2\vartheta}{(1+\rho^2+2\rho\cos 2\vartheta)^{\frac{n}{2}-1}}\\ &= \frac{\sin^{n-2}2\vartheta}{((1+\rho)^2-4\rho\sin^2\vartheta)^{\frac{n}{2}-1}}.\end{aligned}$$

This proves our lemma. □

Now, we can prove the following symmetry.

Lemma 2.10. *If* $\tau\in[0,\pi]$, *then*

$$C(\rho\mathbf{e}_1;\ell_{\pi-\tau}) = C(\rho\mathbf{e}_1;\ell_\tau)$$

for all $\rho\in[0,1]$.

Proof. One obtains this relation immediately from the just derived double integral expression for $C(\rho\mathbf{e}_1;\ell_\tau)$. Introduce there $\pi-\tau$, and then the change of variable: $\pi-\varphi$ instead of φ. Particularly, for the expression $\tilde{\mathcal{G}}(\varphi,\tilde\vartheta,\tau)$ we have

$$\begin{aligned}|\tilde{\mathcal{G}}(\pi-\varphi,\tilde\vartheta,\pi-\tau)| &= \frac{n}{2}|(\cos\tilde\vartheta-\alpha_\rho)\cos(\pi-\tau)+\sin\tilde\vartheta\cos(\pi-\varphi)\sin(\pi-\tau)|\\ &= \frac{n}{2}|-(\cos\tilde\vartheta-\alpha_\rho)\cos\tau-\sin\tilde\vartheta\cos\varphi\sin\tau| = |\tilde{\mathcal{G}}(\varphi,\tilde\vartheta,\tau)|.\end{aligned}$$

It follows the statement of this lemma. □

2.3. **An integral equal to zero.** Our next aim is to show that if we delete the parentheses for absolute value in the integral representation of $C(\rho\mathbf{e}_1;\ell_\tau)$ given in Lemma 2.9, then the double integral obtained in this way is equal to zero. This is a crucial fact in obtaining the final integral representations of the coefficients $C(x;\ell)$ in the next subsection. To prove that we use some facts concerning the Gauss hypergeometric function. For these facts which will be stated below we refer to Chapter 2 and Chapter 3 in [2].

The Gauss hypergeometric function is defined by the hypergeometric series

$${}_2F_1\left(\begin{matrix}a & b\\ & c\end{matrix};z\right) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

for $z\in\mathbf{B}^2$, and by continuation elsewhere. Here, $(a)_k$ stands for the Pochhammer symbol which is defined as

$$(a)_k = \begin{cases} a(a+1)\cdots(a+k-1), & \text{if } k\geq 1,\\ 1, & \text{if } k=0\end{cases}$$

for every complex number a. The series terminates if either a or b is a non–positive integer, in which case the function reduces to a polynomial. For example, if $b=-m$, then

$${}_2F_1\left(\begin{matrix}a & -m\\ & c\end{matrix};z\right) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(a)_k}{(c)_k}z^k.$$

One of the fundamental relations is the Euler integral representation formula

$${}_2F_1\left(\begin{matrix}a & b\\ & c\end{matrix};z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

valid for $\Re c > \Re b > 0$ and $z \neq 1$, $|\arg(1-z)| < \pi$.

One form of the quadratic transform formula states that

$$ {}_2F_1\left(\begin{matrix} a/2 \;\; (a+1)/2 \\ a-b+1 \end{matrix}; z\right) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-a} {}_2F_1\left(\begin{matrix} a \;\; b \\ a-b+1 \end{matrix}; \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right) $$

for $z \in \mathbf{B}^2$.

Lemma 2.11. *The equality*

$$ \int_0^\pi \sin^{n-3}\varphi d\varphi \int_0^{\frac{\pi}{2}} \mathcal{G}(\varphi,\vartheta,\tau)\mathcal{S}(\vartheta)d\vartheta = 0 $$

is valid for every $\tau \in [0, 2\pi]$.

Proof. We have to prove that

$$ \int_0^\pi \sin^{n-3}\varphi d\varphi \int_0^{\frac{\pi}{2}} ((n\cos^2\vartheta - \beta_\rho)\cos\tau + (n\sin\vartheta\cos\vartheta\cos\varphi)\sin\tau)\mathcal{S}(\vartheta)d\vartheta = 0. $$

Since

$$ \int_0^\pi \sin^{n-3}\varphi\cos\varphi d\varphi = 0, $$

we have

$$ \int_0^\pi \sin^{n-3}\varphi d\varphi \int_0^{\frac{\pi}{2}} (n\sin\vartheta\cos\vartheta\cos\varphi)\mathcal{S}(\vartheta)d\vartheta = 0. $$

It remains to consider the integral expression

$$ \int_0^\pi \sin^{n-3}\varphi d\varphi \int_0^{\frac{\pi}{2}} (n\cos^2\vartheta - \beta_\rho)\mathcal{S}(\vartheta)d\vartheta = \frac{\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}J, $$

where

$$ J = \int_0^{\frac{\pi}{2}} (n\cos^2\vartheta - \beta_\rho)\mathcal{S}(\vartheta)d\vartheta. $$

We will prove that $J = 0$, i.e., that $J_1 = J_2$, where

$$ J_1 = n\int_0^{\frac{\pi}{2}} \cos^2\vartheta\mathcal{S}(\vartheta)d\vartheta, \quad J_2 = \beta_\rho\int_0^{\frac{\pi}{2}} \mathcal{S}(\vartheta)d\vartheta. $$

In the integral J_1 introduce the change of variable

$$ \vartheta = \arcsin\sqrt{t}, \quad d\vartheta = \frac{1}{2}t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt. $$

We obtain

$$ \begin{aligned} J_1 &= n\int_0^{\frac{\pi}{2}} \cos^2\vartheta\mathcal{S}(\vartheta)d\vartheta = 2^{n-2}n\int_0^{\frac{\pi}{2}} \frac{\sin^{n-2}\vartheta\cos^n\vartheta}{((1+\rho)^2 - 4\rho\sin^2\vartheta)^{\frac{n}{2}-1}}d\vartheta \\ &= 2^{n-3}n\int_0^1 \frac{t^{\frac{n-3}{2}}(1-t)^{\frac{n-1}{2}}}{((1+\rho)^2 - 4\rho t)^{\frac{n}{2}-1}}dt \end{aligned} $$

Let

$$ \tilde{\rho} = \frac{4\rho}{(1+\rho)^2}. $$

Note that

$$ \sqrt{1-\tilde{\rho}} = \frac{1-\rho}{1+\rho}. $$

By the Euler integral formula it follows

$$
\begin{aligned}
J_1 &= \frac{2^{n-3}n}{(1+\rho)^{n-2}}\int_0^1 t^{\frac{n-3}{2}}(1-t)^{\frac{n-1}{2}}(1-\tilde{\rho}t)^{1-\frac{n}{2}}dt \\
&= \frac{2^{n-3}n}{(1+\rho)^{n-2}}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n)}{}_2F_1\begin{pmatrix} n/2-1 & (n-1)/2 \\ & n \end{pmatrix}; \tilde{\rho}\Big).
\end{aligned}
$$

Using now the quadratic transform formula we obtain

$$
\begin{aligned}
J_1 &= \frac{2^{n-3}n}{(1+\rho)^{n-2}}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n)}(1+\rho)^{n-2}{}_2F_1\left(\begin{matrix} n-2 & -1 \\ & n \end{matrix}; \rho\right) \\
&= 2^{n-3}n\frac{n-1}{2}\frac{\Gamma^2\left(\frac{n-1}{2}\right)}{(n-1)\Gamma(n-1)}\frac{n-(n-2)\rho}{n} \\
&= 2^{n-3}\frac{\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma(n-1)}\beta_\rho.
\end{aligned}
$$

Calculation of J_2 goes in the same way. We have

$$
\begin{aligned}
J_2 &= \beta_\rho\int_0^{\frac{\pi}{2}}\mathcal{S}(\vartheta)d\vartheta = 2^{n-2}\beta_\rho\int_0^{\frac{\pi}{2}}\frac{\sin^{n-2}\vartheta\cos^{n-2}\vartheta}{((1+\rho)^2-4\rho\sin^2\vartheta)^{\frac{n}{2}-1}}d\vartheta \\
&= 2^{n-3}\beta_\rho\int_0^1\frac{t^{\frac{n-3}{2}}(1-t)^{\frac{n-3}{2}}}{((1+\rho)^2-4\rho t)^{\frac{n}{2}-1}}dt.
\end{aligned}
$$

Using the Euler integral formula and then the quadratic transform formula we calculate

$$
\begin{aligned}
J_2 &= \frac{2^{n-3}\beta_\rho}{(1+\rho)^{n-2}}\int_0^1 t^{\frac{n-3}{2}}(1-t)^{\frac{n-3}{2}}(1-\tilde{\rho}t)^{1-\frac{n}{2}}dt \\
&= \frac{2^{n-3}\beta_\rho}{(1+\rho)^{n-2}}\frac{\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma(n-1)}{}_2F_1\left(\begin{matrix} n/2-1 & (n-1)/2 \\ & n-1 \end{matrix}; \tilde{\rho}\right) \\
&= \frac{2^{n-3}\beta_\rho}{(1+\rho)^{n-2}}\frac{\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma(n-1)}(1+\rho)^{n-2}{}_2F_1\left(\begin{matrix} n-2 & 0 \\ & n-1 \end{matrix}; \rho\right) \\
&= 2^{n-3}\beta_\rho\frac{\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma(n-1)}.
\end{aligned}
$$

Therefore, $J_1 = J_2$ which we aimed to prove. □

2.4. **The main representation theorem.** Based on the preceding auxiliary results our aim now is to find representation of $C(x;\ell)$ which is convenient for the extremal problem consideration. This is contained in the next theorem which is the main result of this section. We specify gradient estimates in the normal and tangential directions.

Recall that we have already introduced the following two parameters

$$
\alpha_\rho = \frac{(n-2)\rho}{n}, \quad \beta_\rho = \frac{n-(n-2)\rho}{2}.
$$

Theorem 2.12. *Let $x \in \mathbf{B}^n$ and $\ell \in \partial\mathbf{B}^n$. Denote by τ the angle between the straight lines $N = \{\lambda \mathbf{n}_x : \lambda \in \mathbf{R}\}$ and $L = \{\mu\ell : \mu \in \mathbf{R}\}$. The optimal coefficient $C(x;\ell)$ for the estimate*

$$
\left|\frac{\partial U(x)}{\partial \ell}\right| \le C(x;\ell)(1-|x|)^{-1}\sup_{y\in\mathbf{B}^n}|U(y)|,
$$

where $U(y)$ is among bounded harmonic functions in $\mathbf{B}^n$, may be expressed as follows.

i) If $0 \le \tau < \pi/2$*, then*

$$C(x;\ell) = \frac{4\omega_{n-2}}{\omega_n} \frac{2^{n-1}}{(1+|x|)^{n-1}} \frac{1}{\sqrt{1+\gamma^2}} \int_0^1 \frac{\mathcal{P}_{|x|}(\gamma t) + \mathcal{P}_{|x|}(-\gamma t)}{\sqrt{(1-t^2)^{4-n}}} dt, \quad \gamma = \tan\tau,$$

where the function $\mathcal{P}_\rho(z)$ *of a real argument is defined as*

$$\mathcal{P}_\rho(z) = \int_0^{\frac{z+\sqrt{z^2+1-\alpha_\rho^2}}{1-\alpha_\rho}} \frac{(n-\beta_\rho + nzw - \beta_\rho w^2) w^{n-2} dw}{(1+w^2)^{\frac{n}{2}+1}(1+\kappa_\rho^2 w^2)^{\frac{n}{2}-1}}, \quad \kappa_\rho = \frac{1-\rho}{1+\rho}$$

for $0 \le \rho \le 1$*.*

ii) The coefficient for the pointwise sharp gradient estimate in the radial direction is given by

$$C(x;\mathbf{n}_x) = \frac{4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{2^{n-1}}{(1+|x|)^{n-1}} \int_0^{w_{|x|}} \frac{\beta_{|x|}(w_{|x|}^2 - w^2) w^{n-2} dw}{(1+w^2)^{\frac{n}{2}+1}(1+\kappa_{|x|}^2 w^2)^{\frac{n}{2}-1}},$$

where $w_\rho = \sqrt{\frac{n+(n-2)\rho}{n-(n-2)\rho}}$*.*

iii) Let $\mathbf{t}_x$ *be a tangential direction, i.e., any direction orthogonal to* $\mathbf{n}_x$*. Then*

$$C(x;\mathbf{t}_x) = \frac{2n}{\pi} \frac{2^{n-1}}{(1+|x|)^{n-1}} \int_0^\infty \frac{w^{n-1} dw}{(1+w^2)^{\frac{n}{2}+1}(1+\kappa_{|x|}^2 w^2)^{\frac{n}{2}-1}}.$$

Remark 2.13. Other representations of the coefficient in the pointwise sharp estimate for the absolute value of the radial derivative for a bounded harmonic function in the unit ball were obtained in [11] (see Proposition 4.1 and Corollary 5.4).

Remark 2.14. For $x = 0$ we have $w_0 = 1$, $\beta_0 = n-2$, and $\kappa_0 = 1$. One finds

$$\int_0^{w_0} \frac{\beta_0(w_0^2 - w^2) w^{n-2} dw}{(1+w^2)^{\frac{n}{2}+1}(1+\kappa_0^2 w^2)^{\frac{n}{2}-1}} = \frac{n}{2} \int_0^1 \frac{(1-w^2) w^{n-2}}{(1+w^2)^n} dw$$
$$= \frac{1}{2^n} \frac{n}{n-1}.$$

Since $C(0;\ell)$ is independent of $\ell \in \partial\mathbf{B}^n$ (Remark 2.2), from the part ii) of our theorem we have the following explicit sharp gradient estimate at zero

$$|\nabla U(0)| \le \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \sup_{y\in\mathbf{B}^n} |U(y)|. \tag{2.4}$$

This estimate is well known, it may be found, for example, in Theorem 6.2.6 in [3], or in a more general setting in Corollary 3.2 in [11].

Remark 2.15. The above gradient estimate in zero may be used to obtain the smallest constant C (independent of $x \in \mathbf{B}^n$) such that

$$|\nabla U(x)| \le C(1-|x|)^{-1} \sup_{y\in\mathbf{B}^n} |U(y)|.$$

It is clear that $C = \sup_{x\in\mathbf{B}^n,\,\ell\in\partial\mathbf{B}^n} C(x;\ell)$, but is seems that it is not an easy task to find this extremum. However, we can act as follows. Let $U(y)$ be bounded harmonic function in $\mathbf{B}^n$. For $x \in \mathbf{B}^n$ consider the new harmonic function $V(y) = U(x + (1-|x|)y)$ for $y \in \mathbf{B}^n$. Then

$$|\nabla V(0)| = (1-|x|)|\nabla U(x)|.$$

Since $\sup_{y\in\mathbf{B}^n}|V(y)| \le \sup_{x\in\mathbf{B}^n}|U(x)|$, by applying (2.4) for $V(y)$ we obtain

$$|\nabla U(x)| \le \frac{2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\frac{1}{1-|x|}\sup_{y\in\mathbf{B}^n}|U(y)|. \tag{2.5}$$

Clearly, this estimate is not pointwise sharp, but it is sharp if we take into consideration the whole domain. The estimate (2.5) may be found in [18] on page 139. For such type inequalities we refer to the paper [15] where similar optimal inequalities are considered for more general domains.

We will use the following simple observation concerning the integral of a measurable function. Assume that $\phi(x)$ is a real–valued integrable function on a measure space (X,μ). Let $\phi^+(x) = \max\{\phi(x),0\}$, and $\phi^-(x) = \max\{-\phi(x),0\}$. Denote $\int_X \phi(x)d\mu(x) = J$. Since $\int_X(\phi^+(x)-\phi^-(x))d\mu(x) = J$, we have $\int_X \phi^-(x)d\mu(x) = \int_X \phi^+(x)d\mu(x) - J$, which implies that

$$\int_X |\phi(x)|d\mu(x) = \int_X(\phi^+(x)+\phi^-(x))d\mu(x) = 2\int_X \phi^+(x)d\mu(x) - J.$$

We need the above equality for $J=0$.

Proof of Theorem 2.12. Since $C(x;\ell) = C(x;-\ell)$, regarding Lemma 2.1, Lemma 2.7, and Lemma 2.10, one sees that $C(x;\ell)$, as a function of two variables, depends only on $|x|$ and the angle τ between the straight lines $N = \{\lambda\mathbf{n}_x : \lambda\in\mathbf{R}\}$ and $L = \{\mu\ell : \mu\in\mathbf{R}\}$. Therefore, $C(x;\ell) = C(|x|\mathbf{e}_1;\ell_\tau)$; recall that $\ell_\tau = \cos\tau\mathbf{e}_1 + \sin\tau\mathbf{e}_2$. It follows that in the sequel we should consider the numbers $C(\rho\mathbf{e}_1;\ell_\tau)$ for $0\le\rho\le1$ and $0\le\tau\le\pi/2$. Therefore, we will continue from the integral expression for $C(\rho\mathbf{e}_1;\ell_\tau)$ contained in Lemma 2.9. Recall that we obtained there the following

$$C(\rho\mathbf{e}_1;\ell_\tau) = \frac{4\omega_{n-2}}{\omega_n}\frac{1}{1+\rho}\int_0^\pi \sin^{n-3}\varphi d\varphi\int_0^{\frac{\pi}{2}}|\mathcal{G}(\varphi,\vartheta,\tau)|\mathcal{S}(\vartheta)d\vartheta,$$

where

$$\mathcal{G}(\varphi,\vartheta,\tau) = (n\cos^2\vartheta-\beta_\rho)\cos\tau + (n\sin\vartheta\cos\vartheta\cos\varphi)\sin\tau,$$

and

$$\mathcal{S}(\vartheta) = \frac{\sin^{n-2}2\vartheta}{((1+\rho)^2-4\rho\sin^2\vartheta)^{\frac{n}{2}-1}}.$$

In the above integral expression we introduce the change of variables

$$\vartheta = \arctan w \quad\text{and}\quad \varphi = \arccos t.$$

Then we have

$$\sin\vartheta = \frac{w}{\sqrt{1+w^2}},\quad \cos\vartheta = \frac{1}{\sqrt{1+w^2}},\quad d\vartheta = \frac{dw}{1+w^2}.$$

It follows

$$\begin{aligned}\mathcal{G}(\varphi,\vartheta,\tau) &= (n\cos^2\vartheta-\beta_\rho)\cos\tau + (n\sin\vartheta\cos\vartheta\cos\varphi)\sin\tau\\ &= \frac{(n-\beta_\rho)\cos\tau + (nt\sin\tau)w - (\beta_\rho\cos\tau)w^2}{1+w^2},\end{aligned}$$

and after very short calculation

$$\mathcal{S}(\vartheta) = \frac{2^{n-2}\sin^{n-2}\vartheta\cos^{n-2}\vartheta}{((1+\rho)^2-4\rho\sin^2\vartheta)^{\frac{n}{2}-1}} = \frac{2^{n-2}}{(1+\rho)^{n-2}}\frac{w^{n-2}}{(1+w^2)^{\frac{n}{2}-1}}\frac{1}{(1+\kappa_\rho^2w^2)^{\frac{n}{2}-1}}.$$

If $0 \le \tau < \pi/2$, it is convenient to introduce a new parameter $\gamma = \tan\tau$. Then

$$\cos\tau = \frac{1}{\sqrt{1+\gamma^2}}, \quad \sin\tau = \frac{\gamma}{\sqrt{1+\gamma^2}}.$$

Therefore,

$$\mathcal{G}(\varphi,\vartheta,\tau) = \frac{1}{\sqrt{1+\gamma^2}} \frac{n-\beta_\rho + n\gamma t w - \beta_\rho w^2}{1+w^2}.$$

Altogether we have

$$\begin{aligned}\mathcal{G}(\varphi,\vartheta,\tau)\mathcal{S}(\vartheta)d\vartheta &= \frac{2^{n-2}}{(1+\rho)^{n-2}} \frac{(n-\beta_\rho + n\gamma t w - \beta_\rho w^2)w^{n-2}}{(1+w^2)^{\frac{n}{2}+1}(1+\kappa_\rho^2 w^2)^{\frac{n}{2}-1}}dw \\ &= \frac{2^{n-2}}{(1+\rho)^{n-2}} Q(w)W(w)dw.\end{aligned}$$

The unique positive zero of the quadratic expression

$$Q(w) = n-\beta_\rho + nzw - \beta_\rho w^2$$

is

$$Z(z) = \frac{z+\sqrt{z^2+1-\alpha_\rho^2}}{1-\alpha_\rho}.$$

Regarding the last expression for $C(\rho\mathbf{e}_1;\ell_\tau)$ stated in the beginning of this proof and the observation given before this proof we have

$$\begin{aligned}C(\rho\mathbf{e}_1;\ell_\tau) &= \frac{4\omega_{n-2}}{\omega_n}\frac{2^{n-2}}{(1+\rho)^{n-1}}\frac{1}{\sqrt{1+\gamma^2}}\int_{-1}^{1}\frac{dt}{\sqrt{(1-t^2)^{4-n}}}\int_0^\infty Q(w)W(w)dw \\ &= \frac{4\omega_{n-2}}{\omega_n}\frac{2^{n-1}}{(1+\rho)^{n-1}}\frac{1}{\sqrt{1+\gamma^2}}\int_{-1}^{1}\frac{dt}{\sqrt{(1-t^2)^{4-n}}}\int_0^{Z(\gamma t)} Q(w)W(w)dw.\end{aligned}$$

Finally, applying one more obvious integral transform we obtain the following representation

$$C(\rho\mathbf{e}_1;\ell_\tau) = \frac{4\omega_{n-2}}{\omega_n}\frac{2^{n-1}}{(1+\rho)^{n-1}}\frac{1}{\sqrt{1+\gamma^2}}\int_0^1 \frac{\mathcal{P}_\rho(\gamma t)+\mathcal{P}_\rho(-\gamma t)}{\sqrt{(1-t^2)^{4-n}}}dt,$$

where

$$\mathcal{P}_\rho(z) = \int_0^{Z(z)} Q(w)W(w)dw.$$

This is which we wanted to prove in i).

ii) We have $C(x;\mathbf{n}_x) = C(|x|\mathbf{e}_1;\mathbf{e}_1)$ (because $\mathbf{n}_{\rho\mathbf{e}_1} = \mathbf{e}_1$). The result of this part is contained in i) for $\gamma = 0$. Since

$$\frac{\omega_{n-2}}{\omega_n} = \frac{1}{\pi}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)}, \quad \int_0^1 \frac{dt}{\sqrt{(1-t^2)^{4-n}}} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)},$$

we have

$$\begin{aligned}C(\rho\mathbf{e}_1;\mathbf{e_1}) &= \frac{4\omega_{n-2}}{\omega_n}\frac{2^{n-1}}{(1+\rho)^{n-1}}2\mathcal{P}_\rho(0)\int_0^1\frac{dt}{\sqrt{(1-t^2)^{4-n}}} \\ &= \frac{4}{\sqrt{\pi}}\frac{2^{n-1}}{(1+\rho)^{n-1}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\mathcal{P}_\rho(0),\end{aligned}$$

where

$$\mathcal{P}_\rho(0) = \int_0^{\frac{\sqrt{1-\alpha_\rho^2}}{1-\alpha_\rho}} \frac{(n-\beta_\rho-\beta_\rho w^2)w^{n-2}dw}{(1+w^2)^{\frac{n}{2}+1}(1+\kappa_\rho^2 w^2)^{\frac{n}{2}-1}}.$$

Since $\alpha_\rho = \rho(n-2)/n$, it is easy to obtain $\sqrt{1-\alpha_\rho^2}/(1-\alpha_\rho) = w_\rho$ and $(n-\beta_\rho)/\beta_\rho = w_\rho^2$, which implies the representation of $C(x;\mathbf{n}_x)$ given in this lemma.

iii) For $\tau = \pi/2$ we have

$$\begin{aligned} &C(\rho\mathbf{e}_1;\mathbf{t}_{\rho\mathbf{e}_1}) \\ &= \frac{4\omega_{n-2}}{\omega_n}\frac{1}{1+\rho}\int_0^\pi \sin^{n-3}\varphi d\varphi \int_0^{\frac{\pi}{2}} |\mathcal{G}(\vartheta,\varphi,\gamma)|\mathcal{S}(\vartheta)d\vartheta \\ &= \frac{4\omega_{n-2}}{\omega_n}\frac{2^{n-2}}{(1+\rho)^{n-1}}\int_{-1}^1 \frac{|nt|dt}{\sqrt{(1-t^2)^{4-n}}}\int_0^\infty \frac{w^{n-1}dw}{(1+w^2)^{\frac{n}{2}+1}(1+\kappa_\rho^2 w^2)^{\frac{n}{2}-1}}. \end{aligned}$$

Since

$$\int_{-1}^1 \frac{|nt|dt}{\sqrt{(1-t^2)^{4-n}}} = 2n\int_0^1 \frac{tdt}{\sqrt{(1-t^2)^{4-n}}} = \frac{2n}{n-2},$$

by straightforward calculations we find

$$C(\rho\mathbf{e}_1;\mathbf{t}_{\rho\mathbf{e}_1}) = \frac{2n}{\pi}\frac{2^{n-1}}{(1+\rho)^{n-1}}\int_0^\infty \frac{w^{n-1}dw}{(1+w^2)^{\frac{n}{2}+1}(1+\kappa_\rho^2 w^2)^{\frac{n}{2}-1}}.$$

□

2.5. **Gradient estimates in $\mathbf{R}^n_+$.** In Lemma 2.9 take $\rho = 1$. Then we have $\beta_1 = 1$ and

$$\begin{aligned} \mathcal{G}(\varphi,\vartheta,\tau) &= (n\cos^2\vartheta - 1)\cos\tau + (n\sin\vartheta\cos\vartheta\cos\varphi)\sin\tau. \\ &= \frac{(n\cos^2\vartheta-1)+n\gamma\sin\vartheta\cos\vartheta\cos\varphi}{\sqrt{1+\gamma^2}}, \quad \gamma = \tan\tau. \end{aligned}$$

After a short calculation one finds

$$\mathcal{S}(\vartheta) = \sin^{n-2}\vartheta.$$

It follows that

$$C(\mathbf{e}_1;\ell_\tau) = \frac{2\omega_{n-2}}{\omega_n}\frac{1}{\sqrt{1+\gamma^2}}\int_0^\pi \sin^{n-3}\varphi d\varphi \int_0^{\pi/2} |\mathcal{G}(\varphi,\vartheta,\gamma)|\sin^{n-2}\vartheta d\vartheta,$$

where this time we use the notation $\mathcal{G}(\varphi,\vartheta,\gamma)$ for

$$\mathcal{G}(\varphi,\vartheta,\gamma) = (n\cos^2\vartheta-1)+n\gamma\sin\vartheta\cos\vartheta\cos\varphi,$$

as in [12]. The numbers $C(\ell_\tau) := C(\mathbf{e}_1;\ell_\tau)$ are involved in the optimal gradient estimates of bounded harmonic functions in the half–space $\mathbf{R}^n_+$ in appropriate directions, as G. Kresin and V. Maz'ya proved.

Regarding the above remark we will find $\mathcal{P}_1(z)$ from Theorem 2.12. It is possible to obtain an integral–free expression for $\mathcal{P}_1(z)$. For $\rho = 1$ we have $\alpha_1 = (n-2)/n$, $\beta_1 = 1$ and $\kappa_1 = 0$. Therefore, since

$$\int \frac{(n-1+nzw-w^2)w^{n-2}dw}{(1+w^2)^{\frac{n}{2}+1}} = \frac{w^{n-1}(1+zw)}{(1+w^2)^{\frac{n}{2}}},$$

we have

$$\mathcal{P}_1(z) = \int_0^{\frac{z+\sqrt{z^2+1-\alpha_1^2}}{1-\alpha_1}} \frac{(n-1+nzw-w^2)w^{n-2}dw}{(1+w^2)^{\frac{n}{2}+1}} = \frac{Z^{n-1}(1+zZ)}{(1+Z^2)^{\frac{n}{2}}},$$

where we have denoted

$$Z = Z(z) = \frac{z+\sqrt{z^2+1-\alpha_1^2}}{1-\alpha_1}; \quad \text{then} \quad z = z(Z) = \frac{1-n+Z^2}{nZ}.$$

Since

$$1+zZ = 1+\frac{1-n+Z^2}{nZ}Z = 1+\frac{1-n+Z^2}{n} = \frac{1+Z^2}{n},$$

it follows

$$\mathcal{P}_1(z) = \frac{Z^{n-1}(1+zZ)}{(1+Z^2)^{\frac{n}{2}}} = \frac{1}{n}\frac{Z^{n-1}}{(1+Z^2)^{\frac{n}{2}-1}}.$$

It is not hard to obtain

$$\mathcal{P}_1(z) = \frac{(n-1)^{(n-1)}}{n}\mathcal{P}(y),$$

where

$$y = \frac{nz}{2\sqrt{n-1}} \text{ and } \mathcal{P}(y) = \frac{(y+\sqrt{y^2+1})^{n-1}}{(1+(n-1)(y+\sqrt{y^2+1})^2)^{\frac{n}{2}-1}}.$$

The following result is obtained in [12] with y and $\mathcal{P}(y)$ instead of z and $\mathcal{P}_1(z)$ in the integral expression, but the formulation which follows is more convenient for us.

Proposition 2.16 (Cf. [12]). *Let $x = (x', x_n) \in \mathbf{R}^n_+$, and let ℓ be a unit vector. Let τ be the angle between the straight lines determined by the vectors $\mathbf{n}_x = -\mathbf{e}_n$ and ℓ.*

The optimal coefficient $\mathcal{C}(x;\ell)$ for the estimate

$$\left|\frac{\partial U(x)}{\partial \ell}\right| \le \mathcal{C}(x;\ell) \sup_{y\in\mathbf{R}^n_+} |U(y)|$$

may be represented as

$$\mathcal{C}(x;\ell) = x_n^{-1} C(\ell),$$

where

$$C(\ell) = \frac{4\omega_{n-2}}{\omega_n}\frac{1}{\sqrt{1+\gamma^2}}\int_0^1 \frac{\mathcal{P}_1(\gamma t)+\mathcal{P}_1(-\gamma t)}{\sqrt{(1-t^2)^{4-n}}}dt, \quad \gamma = \tan\tau,$$

if $0 \le \tau < \pi/2$.

For the estimate in the direction $\mathbf{n}_x$ we have

$$\left|\frac{\partial U(x)}{\partial \mathbf{n}_x}\right| \le \frac{4}{\sqrt{\pi}}\frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\frac{1}{x_n}\sup_{y\in\mathbf{R}^n_+}|U(y)|.$$

If $\mathbf{t}_x \perp \mathbf{n}_n$ is any tangential direction, i.e., a unit vector spanned by $\{\mathbf{e}_j : 1 \le j \le n-1\}$, then

$$\left|\frac{\partial U(x)}{\partial \mathbf{t}_x}\right| \le \frac{2}{\pi}\frac{1}{x_n}\sup_{y\in\mathbf{R}^n_+}|U(y)|.$$

($U(y)$ is among bounded harmonic functions in $\mathbf{R}^n_+$)

Remark 2.17. Note that the estimate given in iii) resembles the inequality for harmonic functions in the upper half–plane which is stated in the beginning of the paper. As in [17] it may be proved that the inequality is equivalent to the following one $d_e(U(x), U(y)) \leq 2/\pi d_h(x, y)$ for x, $y \in \partial(\mathbf{R}^n_+ + (0, a))$, where $a > 0$ is any number. Here d_e stands for the Euclidean distance, and d_h is the hyperbolic distance in $\mathbf{R}^n_+$ given by

$$d_h(x, y) = \inf_\gamma \int_\gamma \rho(\omega)|d\omega|$$

(infimum is taken over all rectifiable curves connecting x and y), where $\rho(\omega) = \omega_n^{-1}$, $\omega \in \mathbf{R}^n_+$.

The following corollary gives a connection between the coefficients in the two settings.

Corollary 2.18. *For every $\zeta \in \partial\mathbf{B}^n$ and $\ell \in \partial\mathbf{B}^n$ we have*

$$\lim_{x\to\zeta} C(x; \ell) = C(\ell'),$$

where ℓ' is a such direction that the angle between ζ and ℓ is the same as the angle between ℓ' and $-\mathbf{e}_n$.

3. Partial solution to the optimisation problem

3.1. **Remarks on the optimisation problem.** In this section we consider the optimisation problem

$$\sup_{\gamma\geq 0} C(\rho\mathbf{e}_1; \ell_\tau),$$

for $0 < \rho \leq 1$, where $\gamma = \tan\tau$ (we set $\tan\pi/2 = \infty$). According to the results of the previous section, the Khavinson conjecture for the unit ball is equivalent to the statement that this problem has a solution at $\gamma = 0$ for every $\rho \in (0, 1)$. Our aim here is to prove this statement for ρ enough close to 1. The approach given here is based on the work of G. Kresin and V. Maz'ya [12] where they proved that the optimization problem has a solution at $\gamma = 0$ for $\rho = 1$, which result is equivalent to the Khavinson type problem in the half–space setting.

Regarding Theorem 2.12 and Proposition 2.16 the optimisation problem we consider is equivalent to the following one

$$\sup_{\gamma\geq 0} \frac{1}{\sqrt{1+\gamma^2}} \int_0^1 \frac{\mathcal{P}_\rho(\gamma t) + \mathcal{P}_\rho(-\gamma t)}{\sqrt{(1-t^2)^{4-n}}} dt, \tag{3.1}$$

where the main role is played by the function $\mathcal{P}_\rho(z)$ $(0 < \rho \leq 1)$. Note the following fact. If we are able to establish an inequality of the type

$$\mathcal{P}_\rho(z) + \mathcal{P}_\rho(-z) \leq A(z) \ \text{ for } \ z \in \mathbf{R},$$

together with the equality

$$2\mathcal{P}_\rho(0) = A(0),$$

where $A(z)$ is a non–negative symmetric function, i.e., $A(z) = A(-z)$, and if the new optimisation problem

$$\sup_{\gamma\geq 0} \frac{1}{\sqrt{1+\gamma^2}} \int_0^1 \frac{A(\gamma t)}{\sqrt{(1-t^2)^{4-n}}} dt \tag{3.2}$$

has a unique solution at $\gamma = 0$, then the same is true for the problem (3.1). Although not explicitly stated, the preceding remark is crucial in resolving the optimisation problem for

$\rho = 1$ in [12], along with an inequality which will be stated below in Proposition 3.2. The lemma which follows is inspired by the approach from the mentioned paper.

3.2. **An auxiliary optimisation problem.** We will solve the problem (3.2) for the function

$$A(z) = \sqrt{az^2 + b}, \quad z \in \mathbf{R}.$$

This result will be very useful in the sequel.

Lemma 3.1. *Assume that a and b are positive numbers. If $a/b \le n-1$, then the optimisation problem*

$$\sup_{\gamma \ge 0} \frac{1}{\sqrt{1+\gamma^2}} \int_0^1 \frac{\sqrt{a(\gamma t)^2 + b}}{\sqrt{(1-t^2)^{4-n}}} dt \tag{3.3}$$

has a solution for $\gamma = 0$. Moreover, if $a/b < n-1$, then $\gamma = 0$ is the unique solution to the problem (3.3)*.*

Proof. Straightforward calculations give the following two relations

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)^{4-n}}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

and

$$\int_0^1 \frac{a(\gamma t)^2 + b}{\sqrt{(1-t^2)^{4-n}}} dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{a\gamma^2 + b(n-1)}{n-1}.$$

Applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
\int_0^1 \frac{\sqrt{a(\gamma t)^2 + b}}{\sqrt{1+\gamma^2}\sqrt{(1-t^2)^{4-n}}} dt &= \frac{1}{\sqrt{1+\gamma^2}} \int_0^1 \frac{1}{\sqrt{(1-t^2)^{2-\frac{n}{2}}}} \frac{\sqrt{a(\gamma t)^2 + b}}{\sqrt{(1-t^2)^{2-\frac{n}{2}}}} dt \\
&\le \frac{1}{\sqrt{1+\gamma^2}} \sqrt{\int_0^1 \frac{dt}{\sqrt{(1-t^2)^{4-n}}}} \sqrt{\int_0^1 \frac{a(\gamma t)^2 + b}{\sqrt{(1-t^2)^{4-n}}} dt} \\
&= \frac{1}{\sqrt{1+\gamma^2}} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \sqrt{\frac{a\gamma^2 + b(n-1)}{n-1}} \\
&= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \sqrt{\frac{a\gamma^2 + b(n-1)}{(n-1)(1+\gamma^2)}} \\
&\le \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \sqrt{b}
\end{aligned}$$

for every $\gamma \ge 0$. The second inequality above holds since the function

$$g(\gamma) = \sqrt{\frac{a\gamma^2 + b(n-1)}{(n-1)(1+\gamma^2)}}$$

is decreasing in $\gamma \ge 0$, if $a/b \le n-1$. This follows since

$$\frac{d}{d\gamma} g(\gamma) = \frac{(a-(n-1)b)\gamma}{(n-1)(1+\gamma^2)^2 g(\gamma)} \le 0, \quad \gamma \ge 0.$$

On the other hand, for $\gamma = 0$ we have the equality sign everywhere in the above sequence of estimates.

If $a/b < n-1$, then the extremum is achieved only for $\gamma = 0$, since in this case the function $g(\gamma)$ is strictly decreasing in $\gamma \ge 0$. □

3.3. **An inequality of G. Kresin and V. Maz'ya.** The following nontrivial inequality is established in [12].

Proposition 3.2 (Cf. [12])**.** *Let*

$$\mathcal{P}(y) = \frac{(y+\sqrt{y^2+1})^{n-1}}{(1+(n-1)(y+\sqrt{y^2+1})^2)^{\frac{n}{2}-1}}$$

for $y \in \mathbf{R}$. *Then*

$$\mathcal{P}(y)^2 + \mathcal{P}(-y)^2 \le \frac{4(n-1)(3n-2)y^2 + 2n^2}{n^n}.$$

As a consequence of the above proposition we will derive the following inequality suitable for our needs.

Lemma 3.3. *If* $K = (3n-2)/4$, *then*

$$\mathcal{P}_1(z) + \mathcal{P}_1(-z) \le 2\mathcal{P}_1(0)\sqrt{Kz^2+1}.$$

The inequality is strict, unless for $z = 0$.

Note that $K < n-1$. Actually, the inequality given in the lemma was established in [12] in a different form (via the function $\mathcal{P}(y)$). See the inequality (5.17) on the page 438 there. For the sake of completeness we will write a proof of this fact.

Proof of Lemma 3.3. Recall first that we have

$$\mathcal{P}_1(z) = \frac{(n-1)^{(n-1)}}{n}\mathcal{P}(y), \quad y = \frac{nz}{2\sqrt{n-1}}.$$

Note that $\mathcal{P}_1(-z) = \mathcal{P}(-y)$. Therefore, the inequality in this lemma may be rewritten as

$$\mathcal{P}(y) + \mathcal{P}(-y) \le 2\mathcal{P}(0)\sqrt{K\frac{4(n-1)}{n^2}y^2+1}. \tag{3.4}$$

By Proposition 3.2 we have

$$\mathcal{P}(y)^2 + \mathcal{P}(-y)^2 \le \frac{4(n-1)(3n-2)y^2 + 2n^2}{n^n}.$$

On the other hand, obviously, there holds

$$\mathcal{P}(y)\mathcal{P}(-y) = \frac{1}{(4(n-1)y^2+n^2)^{n/2-1}} \le \frac{1}{n^{n-2}}.$$

This inequality is strict except for $y = 0$. Since $\mathcal{P}(0)^2 = n^{2-n}$, it follows that

$$\begin{aligned}(\mathcal{P}(y) + \mathcal{P}(-y))^2 &\le \frac{4(n-1)(3n-2)y^2 + 4n^2}{n^n} \\ &= 4\mathcal{P}(0)^2\left(\frac{3n-2}{4}\frac{4(n-1)}{n^2}y^2+1\right),\end{aligned}$$

which is inequality (3.4) for $K = (3n-2)/4$. □

3.4. **The optimisation problem for $\rho \neq 1$.** Let us first briefly discus the optimisation problem in the case $\rho = 1$. It may be solved via the inequality given in Lemma 3.3 and our Lemma 3.1. Therefore, the problem for $\rho = 1$ has a unique solution at $\gamma = 0$. Although Lemma 3.1 is not proved in [12], the authors solved the problem on the base of the same approach.

As we have observed the inequality

$$\frac{\mathcal{P}_\rho(z) + \mathcal{P}_\rho(-z)}{2\mathcal{P}_\rho(0)} \leq \sqrt{Kz^2 + 1}, \quad z \in \mathbf{R}, \tag{3.5}$$

is valid for $\rho = 1$, if we take $K = (3n-2)/4 < n-1$. The equality sign (when $\rho = 1$) attains only at $z = 0$. Therefore, for every $z > 0$ we have the strict inequality above.

Let $M > 0$ be an arbitrary big number. Our aim is to show that the inequality (3.5) is valid for every $z \in [0, M]$, if ρ is sufficiently near 1. In that approach we first use the following simple

Lemma 3.4. *Let $F(z)$ and $G(z)$ be two C^2-smooth functions defined in a neighborhood of a segment $[0, l]$. If $F(0) = G(0)$, $F'(0) = G'(0) = 0$, and $F''(z) \leq G''(z)$ for all $z \in [0, l]$, then also $F(z) \leq G(z)$ for all $z \in [0, l]$.*

Particularly, if $F''(0) < G''(0)$, then we have $F''(z) \leq G''(z)$ for all $z \in [0, \varepsilon]$, where $\varepsilon > 0$ is sufficiently small.

Proof. Let us consider the function $H(z) = F(z) - G(z)$ for $z \in [0, l]$. We have $H(0) = H'(0) = 0$ and

$$H''(z) \leq 0, \quad z \in [0, l].$$

Therefore, for every $z \in [0, l]$ we obtain

$$H'(z) = H'(z) - H'(0) = \int_0^z H''(w)dw \leq 0.$$

Similarly, we have

$$H(z) = H(z) - H(0) = \int_0^z H'(w)dw \leq 0,$$

which proves the lemma. □

We will prove now that the validity of the inequality (3.5) for $\rho = 1$ implies the inequality (3.5) for $z \in [0, M]$, if ρ is near 1. Denote the left side in (3.5) by $F(z)$ and the right side by $G(z)$. Because of symmetry we have $F'(0) = G'(0) = 0$. Since for $\rho = 1$ the inequality (3.5) holds, we must have $F''(0) \leq G''(0)$. Otherwise, if the reverse inequality $F''(0) > G''(0)$ would be true, then we also have, in view of Lemma 3.4, the reverse inequality in (3.5) for $\rho = 1$ and for some values of z close to 0, which is incorrect. Moreover, we can achieve the strict inequality $F''(0) < G''(0)$ for $\rho = 1$. Indeed, if the equality $F''(0) = G''(0)$ takes place, then we can slightly increase $K < n-1$ so that $F''(0) < G''(0)$. This is possible because

$$G''(z) = \frac{K}{\sqrt{(1 + Kz^2)^3}},$$

and therefore $G''(0) = K$. We assume in the sequel that this is done. Because of continuity the same inequality for the second derivatives remains valid if ρ is near 1. Applying again Lemma 3.4, we conclude that the inequality (3.5) holds for $z \in [0, \varepsilon]$, where ε is sufficiently close to 0. Since we have strict inequality in (3.5) for $z \in [\varepsilon, M]$, if $\rho = 1$, it follows that

this inequality is true if z belongs to the same segment and if ρ is close to 1. Altogether, we have proved (3.5) for $z \in [0, M]$ and ρ close to 1.

For the sake of simplicity introduce the function $c(\rho, \gamma) = C(\rho \mathbf{e}_1; \ell_\tau)$, where $\gamma = \tan \tau$. Let us consider now our optimisation problem. First of all, we have $c(1, 0) > c(1, \infty)$. Because of continuity, this implies that $c(1, 0) > c(1, \gamma)$, if $\gamma \geq M$, where M is big enough. The last inequality implies that $c(\rho, 0) > c(\rho, \gamma)$ for $\gamma \geq M$, if ρ is close to 1. For $z \in [0, M]$ we have validity of (3.5) (if ρ is perhaps closer to 1). In view of Lemma 3.1, this implies that $c(\rho, 0) > c(\rho, z)$, $z \in [0, M]$. Therefore, the optimisation problem has a unique solution at $\gamma = 0$, if ρ is sufficiently close to 1.

Remark 3.5. It seems that the inequality given in Lemma 3.3, which is, in our approach, crucial in resolving the optimisation problem, is not valid for every $\rho \in (0, 1)$ (even for $\rho \in (0, 0.98)$, if $n = 3$). This suggests that the solution to the optimisation problem is much more harder in general.

3.5. **A boundary result.** At the end of this section let us discus one boundary result. We have proved that $\mathcal{C}(x) = \mathcal{C}(x; \mathbf{n}_x)$, if $x \in \mathbf{B}^n$ is near the boundary $\partial \mathbf{B}^n$. It follows that $(1 - |x|)\mathcal{C}(x) = (1 - |x|)\mathcal{C}(x; \mathbf{n}_x)$, which may be rewritten as

$$\sup_{U,\, \|U\|_\infty \leq 1} (1 - |x|) \left| \nabla U(x) \right| = \sup_{U,\, \|U\|_\infty \leq 1} (1 - |x|) \left| \langle \nabla U(x), \mathbf{n}_x \rangle \right|, \quad \text{if} \quad |x| \approx 1.$$

Letting $x \to \zeta \in \partial \mathbf{B}^n$ above, and bearing in mind that the both sides depend only on $|x|$, we derive

$$\begin{aligned}
\lim_{\mathbf{B}^n \ni x \to \zeta} \sup_{U,\, \|U\|_\infty \leq 1} (1 - |x|) \left| \nabla U(x) \right| &= \lim_{\mathbf{B}^n \ni x \to \zeta} \sup_{U,\, \|U\|_\infty \leq 1} (1 - |x|) \left| \langle \nabla U(x), \mathbf{n}_x \rangle \right| \\
&= \lim_{\mathbf{B}^n \ni x \to \zeta} C(x; \mathbf{n}_x) = C(-\mathbf{e}_n) \\
&= \frac{4}{\sqrt{\pi}} \frac{(n-1)^{\frac{n-1}{2}}}{n^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}
\end{aligned}$$

(see Corollary 2.18). This boundary relation is a special case of the corresponding result in [12] obtained for more general domains in $\mathbf{R}^n$.

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